বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examination 2022

(Under CBCS Pattern)

Semester - VI

Subject : MATHEMATICS

Paper : DSE 3 - T

Full Marks : 60

Time : 3 Hours

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

[MECHANICS]

1. Answer any ***five*** questions : 2×5=10

(a) What are stable and unstable equilibrium ? State energy test of stability.

(b) State the principle of virtual work for a system of forces acting on a rigid body.

(c) State Kepler's laws of planetary motion.

(d) If the anglular velocity about the origin be a constant w, then find the cross-radial component of rate of change of acceleration of the particle.

(e) If X, Y be the algebraic sums of the resolved parts of all the forces acting in a plane along the rectangular axes θ_x and θ_y; and G be the algebraic sum of the moments of the forces about O; find the locus of the points at which the algebraic sum of the moments of the forces is constant and is equal to G^1.

(f) Define 'apse' of a central orbit. Show that, at an apse, a particle is moving at right angles to the radius vector of the point.

(g) An artificial satelite revolves about the earth at a height H above the surface. Find the orbital speed, so that a man in the satelite will be in a state of weightlessness.

(h) State D' Alembert's principle. Write down the general equations of motion of a rigid body.

2. Answer any ***four*** questions : 5×4=20

(a) Find the co-ordinates of C.G. of a lamina in the shape of a quadrant of the curve

$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$, density at (x, y) is $\rho = kxy$, where k is constant.

(b) A square lamina rests with its plane perpendicular to a smooth wall, one corner being attached to a point in the wall by a fine string of length equal to the side of the square. Find the position of equilibrium and show that it is stable.

(c) At the vertex C of a triangle ABC which is right angled at C, show that the principal axes are a perpendicular to the plane and two others inclined to the sides at an angle $\frac{1}{2}\tan^{-1}\frac{ab}{a^2-b^2}$.

(d) An ellipse of axes a, b and a circle of radius b are cut from the same sheet of a uniform metal and are suspended and fixed together with their centres coincident. The figure is free to move in its own vertical plane about one end of its major axis. Show that the length of the equivalent simple pendulum is $\frac{5a^2-ab+2b^2}{4a}$.

(e) A particle is projected at right angles to the line joining it to a centre of force, attracting according to the law of inverse square of the distance, with a velocity $\frac{\sqrt{3}}{2}V$, where V denotes the velocity from infinity. Find the eccentricity of the orbit described and show that the periodic time is $2\pi T$; T being the time taken to describe the major-axis of the orbit with velocity V.

(f) Find the accelerations of a particle, moving in 3-dimensional space, in terms of polar co-ordinates.

3. Answer any ***three*** questions : 10×3=30

(a) (i) The middle points of opposite sides of a quadrilateral formed by four freely jointed weightless bars are connected by two light rods of length 'a' and 'b' in a state of tension. If T_1 and T_2 be the tensions of those rods, prove that $\frac{T_1}{a}+\frac{T_2}{b}=0$.

(ii) A surface is formed by revolution of rectangular hyperbola about a vertical asymptote; show that a particle will rest on it everywhere beyond its intersection with a certain circular cylinder. 5+5

(b) (i) If X, Y, Z, L, M, N are six components of a system of forces, deduce the invariants of the system.

(ii) Equal forces act along the axes and along the straight line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$. Find the equation of the central axis of the system.

5+5

(c) (i) A particle moves with a central acceleration $\{\mu \div (\text{distance})^2\}$. It is projected with a velocity V at a distance R. Show that its path is a rectangular hyperbola, if the angle of projection is $\sin^{-1}\left\{\frac{\mu}{VR\sqrt{V^2-\frac{2\mu}{R}}}\right\}$.

(ii) One end of an elastic string, of unstretched length 'a', is tied to a point on a smooth table and a particle is attached to the other end and can move freely on the table. If the path be nearly a circle of radius b, then show that apsidal angle is approximately $\pi\sqrt{\frac{b-a}{4b-3a}}$. 5+5

(d) (i) A thin rod of length $2a$ revolves with uniform angular velocity ω about a vertical axis through a small joint at one extremity of the rod, so that it describes a cone of semi-vertical angle α. Show that $\omega^2=\frac{3g}{4a\cos\alpha}$.

(ii) An elliptic lamina is such that when it swings about one latus rectum as a horizontal axis, the other latus rectum passes through the centre of oscillation. Prove that the eccentricity of the ellipse is $\frac{1}{2}$. 5+5

(e) (i) A particle of Unit mass is projected with velocity u at an inclination α about the horizon in a medium whose resistance is k times the velocity. Show that the direction of the path described will again make an angle α with the horizon after a time $\frac{1}{k}\log\left\{1+\frac{2ku}{g}\sin\alpha\right\}$.

(ii) Find the apsidal angle in a nearly circular orbit under the central force $ar^m + br^n$; a, b are constants. 6+4

OR

[NUMBER THEORY]

1. Answer any ***five*** questions : 2×5=10

(a) If a is odd integer then show that $32 \mid (a^2 + 7)(a^2 + 3)$.

(b) Show that 41 divides $2^{20} - 1$.

(c) If $n > 1$ is an integer not of the form $6k + 3$, prove that $n^2 + 2^n$ is composite.

(d) Let $\tau(n)$ denote the number of positive divisors of n and $\sigma(n)$ denote the sum of these divisors. Find $\tau(160)$ and $\sigma(160)$.

(e) Define Legendre symbol. Find the value of the Legendre symbols 9/23.

(f) If p and q are odd primes satisfying $p = q + 4a$ for some a, establish that $\left(\frac{a}{p}\right) = \left(\frac{a}{q}\right)$.

(g) Let $a_n = 6^n + 8^n$. Determine the remainder on dividing a_{83} by 49.

(h) For what valus of n does $n!$ terminates in 37 zeros?

2. Answer any ***four*** questions : 5×4=20

(a) Prove that the integer $53^{103} + 103^{53}$ is divisible by 39, and that $111^{333} + 333^{111}$ is divisible by 7.

(b) Determine all solutions in the positive integers of the Diophantine equations : $123x + 360y = 99$.

(c) Let $n = p_1^{k_1}, p_2^{k_2} .....p_r^{kr}$ be the prime factorization of the integer $n > 1$. If f is a multiplicative function that is not identically zero, prove that

$$\sum_{d/n} \mu(d) f(d) = (1 - f(p_1))(1 - f(p_2)) ......(1 - f(p_r))$$

(d) If $n \geq 1$ and p is a prime, prove that the exponent of the highest power of p that divides $(2n)!(n!)^2$ is $\sum_{k=1}^{\infty}\left(\left[\frac{2n}{p^k}\right] - 2\left[\frac{n}{p^k}\right]\right)$.

(e) Let p be an odd prime and gcd (a, p) = 1. Then a is a quadratic residue of p if and only if $a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$.

(f) Encrypt the plaintext message GODL MEDAL using the RSA algorithm with key (n, k) = (2419, 3).

3. Answer any ***three*** questions : 10×3=30

(a) (i) Show that all the solutions of the Pythagorean equation $x^2 + y^2 = z^2$ satisfying the conditions $\gcd(x, y, z) = 1, 2 \mid x,\ x > 0, y > 0, z > 0$ are given by the formulas $x = 2st,\ y = s^2 - t^2,\ z = s^2 + t^2$ for integers s > t > 0 such that gcd (s, t) = 1 and $s \neq t \pmod{2}$. 5

(ii) Find all positive integers n such that $3^{n-1} + 5^{n-1} \mid 3^n + 5^n$. 3

(iii) If a primitive root exists for a positive integer n, then determine the exact number of primitive root for n. 2

(b) (i) Let p be an odd prime and let gcd (a, p) = 1. If n denotes the number of integers in the set $S = \left\{ a, 2a, 3a, \ldots\ldots, \left(\frac{p-1}{2}\right)a \right\}$.

whose remainders upon division by exceed $\frac{p}{2}$, then show that $\left(\frac{a}{p}\right) = (-1)^n$.

(ii) If m and n are relatively prime positive integers, prove that $m^{\varphi(n)} + n^{\varphi(m)} \equiv 1$ (mod mn). 5

(c) Given a positive integer n, let $\tau(n)$ denote the number of positive divisors of n and $\sigma(n)$ denote the sum of these divisors. If $n = p_1^{k_1} p_2^{k2} \ldots . p_r^{k_r}$ is the prime factorization of $n > 1$, then prove that

(i) $\tau(n) = (k_1 + 1)(k_2 + 1) \ldots . (k_r + 1)$, and

(ii) $\sigma(n) = \dfrac{p_1^{k_1+1} - 1}{p_1 - 1} \dfrac{p_2^{k_2+1} - 1}{p_2 - 1} \ldots . \dfrac{p_r^{k_r+1} - 1}{p_r - 1}$

(d) Let $n = a_m(1000)^m + a_{m-1}(1000)^{m-1} + ... + a_1(1000) + a_0$ where a_k are integers and $0 \le a_k \le 999, k = 0,1,......., m$ be the representation of a positive integer n.

Let $T = a_0 - a_1 + a_2 - ...... + (-1)^m a_m$. Then

(i) n is divisible by 7 if and only if T is divisible by 7.

(ii) n is divisible by 11 if and only if T is divisible by 11.

(iii) n is divisible by 13 if and only if T is divisible by 13. 3+3+4

(e) (i) Let p be an odd prime and $gcd(a, p) = 1$. Then prove that a is a quadratic residue of p if and only if $a^{\frac{p-1}{2}} \equiv 1 (\bmod\ p)$. 5

(ii) If $p = 2^k + 1$ is prime, verify that every quadratic non residue of p is a primitive root of p. 3

(iii) Find the value of (–72/131). 2

OR

[INDUSTRIAL MATHEMATICS]

1. Answer any ***five*** questions : 2×5=10

   (a) Write the full form of MRI.

   (b) What is the full form of CT?

   (c) Who invented CT scan?

   (d) Mention two limitations of CT.

   (e) What is computed tomography

   (f) What is a scatter radiation?

   (g) Which CT imagers is often referred to as the heart scan?

   (h) Is the radiation from an X-Ray dangerous?

2. Answer any ***four*** questions : 5×4=20

   (a) Discuss various types of medical imaging.

   (b) What is inverse problem in image processing?

   (c) Discuss the concept behind the city scan.

   (d) How image is formed in CT scan?

   (e) Discuss various types of X-Ray machines.

   (f) Discuss the properties of back projection.

3. Answer any ***three*** questions : 10×3=30

   (a) How do you solve inverse problems?

   (b) Define random transform with an example. How is it used to obtain the projections of object? 4+6

   (c) How does iterative reconstruction work in CT?

   (d) Discuss the reconstruction methods for CT imaging.

   (e) Discuss geological anomalies in Earth's interior from measurements at its surface.